Fundamental properties of Dyn. Sys.:

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0$$

- state $x(t) \in \mathbb{R}^n$, $f: [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$
  vector field
- special case:

$$\dot{x}(t) = f(x(t)) \quad \text{- autonomous}$$

- Objective: Specify sufficient condition on f so that a unique solution exists

Ex1:

$$\dot{x} = a x, \quad x(t_0) = x_0$$

$$x \in \mathbb{R}$$

$$\Longrightarrow \underbrace{x(t) = e^{a(t-t_0)} x_0}_{\text{solution exists for all } t}$$

Ex2:

$$\dot{x} = \sqrt{x}, \qquad x(0) = 0$$

Solution 1: $x(t) = 0$

check: $\dot{x}(t) = 0 = \sqrt{x(t)}$ ✓

Solution 2: $x(t) = \dfrac{t^2}{4}$

check: $\dot{x}(t) = \dfrac{t}{2} = \sqrt{\dfrac{t^2}{4}} = \sqrt{x(t)}$ ✓

So solution exists, but not <u>unique</u>

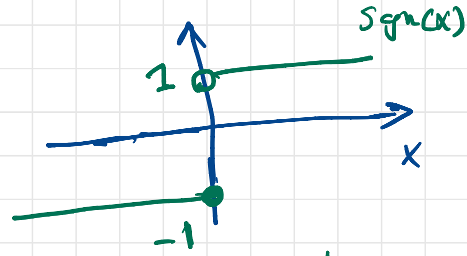
Ex3: $\dot{x} = x^2, \qquad x(0) = x_0 > 0$

$$\Rightarrow \quad x(t) = \frac{1}{\frac{1}{x_0} - t}$$

solution blows up at $t = \dfrac{1}{x_0}$

Ex4:

$$\dot{x} = -\mathrm{sgn}(x), \quad x(0) = 0$$



$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x \le 0 \end{cases}$$

- Argue that a solution can not exist.

$$\dot{x}(0) = -\mathrm{sgn}(0) = 1$$

$\Rightarrow x(t) > 0$ for sufficiently small t

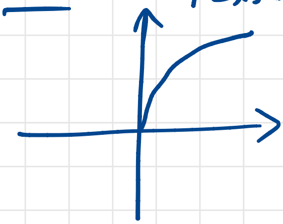
However

$$x(t) = -\int_0^t \underbrace{\mathrm{sgn}(x(s))}_{=1}\, ds < 0$$

Contradiction

- Let's discuss what goes wrong relative to linear

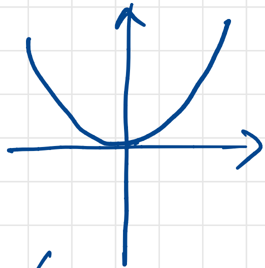
Ex2 $f(x) = \sqrt{x}$



$f'(x) = \frac{1}{2\sqrt{x}} \to \infty$

as $x \to 0$

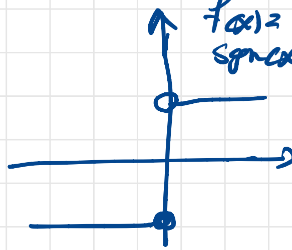
Ex3 $f(x) = x^2$



$f'(x) = 2x \to \infty$

as $x \to \infty$

Ex4 $f(x) = \operatorname{sgn}(x)$



f is not continuous at $x = 0$

$\Rightarrow$ the rate of growth of f affects existence and uniqueness of the solution.

Some definitions:

- $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz if for every compact set $D \subset \mathbb{R}^n$, $\exists\, L > 0$ s.t.

  (compact: closed and bounded)

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad \forall x, y \in D$$

- It can be defined w.r.t different norms. we use Euclidean norm

- f is globally Lipschitz if $\exists L>0$ s.t.

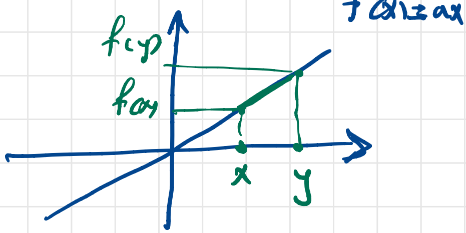
$$\|f(x)-f(y)\| \leq L\|x-y\| \quad \forall x,y \in \mathbb{R}^n$$

- Global Lip. $\Leftrightarrow$ local Lip.

- Equivalently: $\dfrac{\|f(x)-f(y)\|}{\|x-y\|} \leq L \quad \forall x,y$

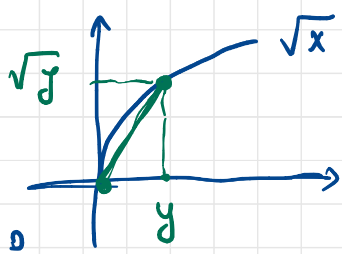
- Ex1:

$$f(x) = ax$$



$$|f(x)-f(y)| = |ax-ay|$$
$$\leq \underbrace{|a|}_{L} |x-y| \Rightarrow \text{Globally Lip.}$$

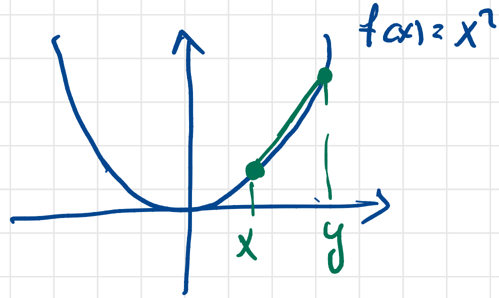
Ex2: $f(x) = \sqrt{x}$



$x=0, y>0,$ $\dfrac{\sqrt{y}-0}{y-0} = \dfrac{1}{\sqrt{y}} \to \infty$ as $y \to 0$

$\Rightarrow$ Not locally Lip.

Ex3 : $f(x) = x^2$



$$\frac{|f(y)-f(x)|}{|y-x|} = \frac{|y^2-x^2|}{|y-x|}$$

$= |y+x|$ is bounded if x, y are bdd.

$\Longrightarrow$ locally Lip.

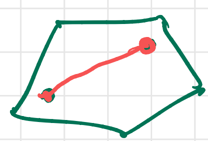
- Observation:
  - if $|f'(x)|$ is bdd on any compact set then, f is Locally Lip.
  - if $|f'(x)|$ is bdd globally $\Rightarrow$ f is globally Lip.

Lemma : (lemma 3.1 in Khalil)

if $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and for convex set W

$$\left\| \frac{\partial f}{\partial x}(x) \right\| \leq L \quad \forall x \in W \Longrightarrow \|f(x)-f(y)\| \leq L\|x-y\| \quad \forall x, y \in W$$

- Convex set: any two points are connected with a line inside the set



Examples:

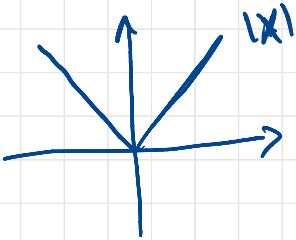
$f(x) = ax \Rightarrow |f'(x)| = |a| \Rightarrow$ globally Lip

$f(x) = \sqrt{x} \Rightarrow |f'(x)| = \frac{1}{\sqrt{x}} \longrightarrow$ Not bdd.

$f(x) = x^2 \Rightarrow |f'(x)| = 2|x| \Rightarrow$ Locally Lip.

- A non-diffrentiable example:

$f(x) = |x|$



$|f(x) - f(y)| = \big||x| - |y|\big|$

$\le |x - y| \Rightarrow$ globally Lip.

- How strong is Lip. assumption?

diffreniable with bdd derivative $\Longrightarrow$ Lip. $\Longrightarrow$ Continuous

- $f(t,x)$ is Lip. in x uniformly in $t \in [a,b]$ if

$$\|f(t,x) - f(t,y)\| \le L\|x-y\| \quad \forall x,y \in D, \forall t \in [a,b]$$

Thm: (Existence and uniqueness)

- Assume
  1. $f(t,x)$ is piecewise continuous in t
  2. Locally Lip. in x, uniformly in $t \in [t_0, t_1]$
- Then, for $x_0 \in \mathbb{R}^n$, $\exists\, \delta > 0$ s.t.
  there exists a unique solution on $[t_0, t_0+\delta]$
  for $\dot{x}(t) = f(t, x(t))$, $x(t_0) = x_0$
- If f is globally Lip. $\Rightarrow$ solution exists for all $t > t_0$